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Improved effective action for light quarks beyond the chiral limit

M. Musakhanov[a]

Theoretical Physics Department, Tashkent State University, Tashkent 700095, Uzbekistan
(e-mail: yousuf@univer.tashkent.su, cc: yousuf@iaph.silk.org)

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Abstract. We propose an improvement of the Diakonov–Petrov effective action on the basis of the Lee–Bardeen results for the quark determinant in the instanton field. This improved effective action provides a proper account of the current quark masses, which is particularly important for strange quarks. This action is successfully tested by calculations of the quark condensate, the masses of the pseudoscalar meson octet and axial-anomaly low-energy theorems.

1 Introduction

Without any doubt instantons are very important components of the QCD vacuum. Their properties are described by the average instanton size ρ and the inter-instanton distance R. In 1982, Shuryak [1] fixed them phenomenologically as

$$\rho = 1/3\,\mathrm{fm}, \quad R = 1\,\mathrm{fm}. \tag{1}$$

The validity of such parameters was confirmed by theoretical variational calculations [2] and recent lattice simulations of the QCD vacuum (see a recent review [3]).

The presence of instantons in QCD vacuum very strongly affects light-quark properties, due to instanton–quark rescattering and the consequent generation of quark–quark interactions.

These effects lead to the formation of the massive constituent interacting quarks. This implies spontaneous breaking of chiral symmetry (SBCS), which leads to the collective massless excitations of the QCD vacuum-pions. The most important degrees of freedom in low-energy QCD are these quasi-particles. Therefore instantons play a leading role in the formation of the lightest hadrons and their interactions, while the confinement forces are probably rather unimportant.

All these properties are concentrated in the effective action in terms of quasi-particles. Effective actions for quarks in the field of the instantons go back to Shifman, Vainshtein and Zakharov [4]. A very successful attempt to construct them was made by Diakonov and Petrov (DP) in 1986 (see recent papers [5] and a recent detailed review [6] and references therein). Starting from the instanton model of QCD vacuum, they postulated the effective action on the basis of the interpolation formula for the well-known expression for the light-quark propagator $S_\pm$ in the field of a single instanton (anti-instanton):

$$S_\pm \approx S_0 + \Phi_{\pm,0}\Phi^\dagger_{\pm,0}/\mathrm{i}m. \tag{2}$$

Here $S_0 = (\mathrm{i}\hat\partial)^{-1}$ and $\Phi_{\pm,0}$ are the quark zero-modes generated by instantons[1]. This approach was extended to the $N_f = 3$ case [7] with the fluctuations of the number of instantons taken into account [8].

On the other hand, Lee and Bardeen [9] (LB) derived the quark propagator in the background of many instantons in a much more sophisticated approximation than DP. It is written as

$$S \approx S_0 + \sum_i^N (S_i^{\mathrm{NZM}} - S_0) + (B^{-1})_{ij} \times \left[|\Phi_{j,0}\rangle + R^+(-m)|\Phi_{j,0}\rangle\right]\left[\langle\Phi_{i,0}| + \langle\Phi_{i,0}|R(m)\right], \tag{3}$$

where

$$B_{ij} = \mathrm{i}m\delta_{ij} + a_{ji}, \tag{4}$$

and a_{ij} is the overlapping matrix element of the quark zero-modes $\Phi_{\pm,0}$ generated by instantons (anti-instantons). This matrix element is non-zero only between instantons and anti-instantons (and vice versa) due to specific chiral properties of the zero-modes and it is equal to

$$a_{-+} = -\langle\Phi_{-,0}|\mathrm{i}\hat\partial|\Phi_{+,0}\rangle. \tag{5}$$

The overlapping of the quark zero-modes provides the propagation of the quarks by jumping from one instanton to another. The quantity

$$R(m) = \sum_i^N g(\hat A - \hat A_i)\left(S_i^{\mathrm{NZM}} - S_0\right) - \mathrm{i}m \times \left[S_0 + \sum_i^N (S_i^{\mathrm{NZM}} - S_0)\right]$$

[a] Associate Member of ICTP.

[1] $\Phi_{\pm,\lambda}$ is the eigen-solution of the Dirac equation $(\mathrm{i}\hat\partial + g\hat A_\pm)\Phi_{\pm,\lambda} = \lambda\Phi_{\pm,\lambda}$ in the instanton (anti-instanton) field $A_{\pm,\mu}(x;\xi_\pm)$.

describes the contribution of the non-normalizable continuum. The importance of this contribution was mentioned in [9].

With this quark propagator (3) the corresponding fermionic determinant in the field of many instantons was calculated by LB, who found an amusing result for this quantity:

$$\det{}_N = \det B, \quad B_{ij} = \mathrm{i}m\delta_{ij} + a_{ji}. \tag{6}$$

So the determinant of the infinite matrix was reduced to the determinant of the finite matrix in the space of *only zero-modes*. From (4), (5) and (6) it is clear that for $N_+ \neq N_-$:

$$\det{}_N \sim m^{|N_+ - N_-|}, \tag{7}$$

which will strongly suppress the fluctuations of $|N_+ - N_-|$. Therefore in final formulas we will assume that $N_+ = N_- = N/2$.

Recently a correction to the result of Lee and Bardeen in the simplest case of an instanton–anti-instanton molecule was proposed [14]. It was shown that the correction has a structure of the type

$$\det B = \left(m^2 (1 + O(\rho^6 R^{-6})) + |a|^2 \left(1 + \frac{a + a^*}{|a|^2} O(\rho^5 R^{-6}) \right) \right). \tag{8}$$

Since $a \sim \rho^2 R^{-3}$, the second addend of the coefficient of $|a|^2$ in (8) is of order $O(\rho^3 R^{-3}) \sim (1/3)^3$; therefore both correction terms which appear in that expression may be neglected. It follows that (6) take the masses of the current quarks properly into account. Here we observe the competition between the current mass m and the overlapping matrix element $a \sim \rho^2 R^{-3}$. With typical instanton sizes $\rho \sim 1/3\,\mathrm{fm}$ and inter-instanton distances $R \sim 1\,\mathrm{fm}$, a is of the order of the strange current quark mass $m_{\mathrm{s}} = 150\,\mathrm{MeV}$. Therefore it is very important in this case to take the current quark mass properly into account.

In our previous papers [10–12] we showed that the constituent quarks appear as effective degrees of freedom in the fermionic representation of $\det B$. This approach led to the DP effective action with a specific choice of these degrees of freedom. This effective action fulfilled some axial-anomaly low-energy theorems in the chiral limit, but failed beyond this limit [10,11], so this action is hardly applicable to the strange quarks.

The aim of this work is to find an improved effective action so as to take current quark masses into account. We essentially follow the same approach as in our previous works [10–12], but with another fermionic representation of $\det B$, which leads to a different choice of the degrees of freedom in the effective action. This improved effective action will be checked against direct calculations of the quark condensate, the masses of the pseudoscalar meson octet and axial-anomaly low-energy theorems beyond the chiral limit.

2 The derivation of the improved effective action

The effective action follows from the fermionic representation of $\det_N$ [5]. This is not a unique operation. The problem is to take a proper representation which will define the main degrees of freedom in low-energy QCD constituent quarks.

Let us rewrite $\det_N$ following the idea suggested in [13]. First, by introducing the Grassmanian (N_+, N_-) vector

$$\Omega = (u_1 \cdots u_{N_+}, v_1 \cdots v_{N_-})$$

and

$$\bar{\Omega} = (\bar{u}_1 \cdots \bar{u}_{N_+}, \bar{v}_1 \cdots \bar{v}_{N_-})$$

we can rewrite

$$\det{}_N = \int \mathrm{d}\Omega \mathrm{d}\bar{\Omega} \exp(\bar{\Omega} B \Omega), \tag{9}$$

where

$$\begin{aligned} \bar{\Omega} B \Omega &= \bar{\Omega}(\mathrm{i}m + a^{\mathrm{T}})\Omega \\ &= \mathrm{i}\sum_{+} m\bar{u}_+ u_+ + \mathrm{i}\sum_{-} m\bar{v}_- v_- \\ &\quad + \sum_{+-} (\bar{u}_+ v_- a_{-+} + \bar{v}_- u_+ a_{+-}). \end{aligned} \tag{10}$$

The product $\bar{u}_+ v_- a_{-+}$ and $\bar{u}_+ \mathrm{i}m u_+, \bar{v}_- \mathrm{i}m v_-$ can be rewritten in the form

$$\begin{aligned} \bar{u}_+ a_{-+} v_- &= -v_- a_{-+} \bar{u}_+ \\ &= +\langle (\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{-,0} v_-| \\ &\quad \times (\mathrm{i}\hat{\partial} + \mathrm{i}m)^{-1} |(\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{+,0}\bar{u}_+\rangle. \end{aligned} \tag{11}$$

$$\begin{aligned} \bar{u}_+ \mathrm{i}m u_+ &= -\langle \Phi_{+,0} u_+ |(\mathrm{i}\hat{\partial} + \mathrm{i}m)| \Phi_{+,0}\bar{u}_+\rangle \\ &= -\langle (\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{+,0} u_+| \\ &\quad \times (\mathrm{i}\hat{\partial} + \mathrm{i}m)^{-1} |(\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{+,0}\bar{u}_+\rangle, \end{aligned} \tag{12}$$

$$\begin{aligned} \bar{v}_- \mathrm{i}m v_- &= -\langle (\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{-,0} v_-| \\ &\quad \times (\mathrm{i}\hat{\partial} + \mathrm{i}m)^{-1} |(\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{-,0}\bar{v}_-\rangle. \end{aligned} \tag{13}$$

The next step is to introduce N_+, N_- sources $\eta = (\eta_+, \eta_-)$ and N_-, N_+ sources $\bar{\eta} = (\bar{\eta}_-, \bar{\eta}_+)$. $\bar{\eta}_-$, $\bar{\eta}_+$, η_+ and η_- are defined as

$$\begin{aligned} \bar{\eta}_- &= \langle (\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{-,0} v_-|, & \bar{\eta}_+ &= \langle (\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{+,0} u_+|, \\ \eta_+ &= |(\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{+,0}\bar{u}_+\rangle, & \eta_- &= |(\mathrm{i}\hat{\partial} + \mathrm{i}m)\Phi_{-,0}\bar{v}_-\rangle. \end{aligned}$$

Then $(\bar{\Omega} B \Omega)$ can be rewritten as

$$\begin{aligned} (\bar{\Omega} B \Omega) &= -\bar{\eta}_+ (\mathrm{i}\hat{\partial} + \mathrm{i}m)^{-1}\eta_+ - \bar{\eta}_- (\mathrm{i}\hat{\partial} + \mathrm{i}m)^{-1}\eta_- \\ &\quad + \bar{\eta}_- (\mathrm{i}\hat{\partial} + \mathrm{i}m)^{-1}\eta_+ + \bar{\eta}_+ (\mathrm{i}\hat{\partial} + \mathrm{i}m)^{-1}\eta_- \end{aligned} \tag{14}$$

and $\det_N$ can be rewritten as

$$\det{}_N = \int \mathrm{d}\Omega \mathrm{d}\bar{\Omega} \exp(\bar{\Omega} B \Omega)$$

$$= \left(\det(\mathrm{i}\hat{\partial}+\mathrm{i}m)\right)^{-1}\int \mathrm{d}\Omega\mathrm{d}\bar{\Omega}\mathrm{D}\psi\mathrm{D}\psi^\dagger \times\exp\int \mathrm{d}x(\psi^\dagger(x)(\mathrm{i}\hat{\partial}+\mathrm{i}m)\psi(x) +\bar{\eta}_+(x)\psi(x)-\bar{\eta}_-(x)\psi(x) +\psi^\dagger(x)\eta_+(x)-\psi^\dagger(x)\eta_-(x)). \quad (15)$$

The integration over Grassmanian variables Ω and $\bar{\Omega}$ (with account taken of the N_f flavors $\det_N=\prod_f \det B_f$) provides the fermionized representation of the result of Lee and Bardeen for $\det_N$ in the form:

$$\det{}_N=\int \mathrm{D}\psi\mathrm{D}\psi^\dagger\exp\left(\int \mathrm{d}^4x\sum_f \psi_f^\dagger(\mathrm{i}\hat{\partial}+\mathrm{i}m_f)\psi_f\right) \times\prod_f\left\{\prod_+^{N_+}V_+[\psi_f^\dagger,\psi_f]\prod_-^{N_-}V_-[\psi_f^\dagger,\psi_f]\right\}, \quad (16)$$

where

$$V_\pm[\psi_f^\dagger,\psi_f]=\int \mathrm{d}^4x\left(\psi_f^\dagger(x)(\mathrm{i}\hat{\partial}+\mathrm{i}m_f)\Phi_{\pm,0}(x;\xi_\pm)\right) \times\int\mathrm{d}^4y\left(\Phi_{\pm,0}^\dagger(y;\xi_\pm)(\mathrm{i}\hat{\partial}+\mathrm{i}m_f)\psi_f(y)\right). \quad (17)$$

Equation (16) exactly represents the fermionic determinant in terms of constituent quarks ψ_f. This expression differs from the ansatz on the fixed N partition function postulated by DP by another way of taking account of the current mass of quarks.

Let us calculate the quark propagator S in the field of instanton–anti-instanton pairs. First we calculate the partition function Z_N:

$$Z_N=\det{}_N=-m^2-|a|^2, \quad (18)$$

where

$$a=-\langle\Phi_{-,0}|\mathrm{i}\partial|\Phi_{+,0}\rangle. \quad (19)$$

Taking (18) and (16) into account, we find the propagator

$$S=(\mathrm{i}\hat{\partial}+\mathrm{i}m)^{-1} -\frac{\mathrm{i}m(\Phi_{+,0}\Phi_{+,0}^\dagger+\Phi_{-,0}\Phi_{-,0}^\dagger)+a\Phi_{-,0}\Phi_{+,0}^\dagger+a^*\Phi_{+,0}\Phi_{-,0}^\dagger}{m^2+|a|^2}. \quad (20)$$

As was mentioned before, with typical instanton sizes ρ and inter-instanton distances R (1), a is of the order of the strange current quark mass $m_\mathrm{s}=150\,\mathrm{MeV}$. Again we conclude that, in this case, it is very important to take the current quark mass into account.

Keeping in mind the low density of the instanton media, which allows independent averaging over positions and orientations of the instantons, (16) we are led to the partition function

$$Z_N=\int\mathrm{D}\psi\mathrm{D}\psi^\dagger\exp\left(\int\mathrm{d}^4x\sum_f\psi_f^\dagger(\mathrm{i}\hat{\partial}+\mathrm{i}m_f)\psi_f\right) \times W_+^{N_+}W_-^{N_-}, \quad (21)$$

where

$$W_\pm=\int\mathrm{d}\xi_\pm\prod_f V_\pm[\psi_f^\dagger\psi_f]. \quad (22)$$

The integral in $\xi_\pm$ for the simplest case $N_f=1$ is

$$\int\mathrm{d}\xi_\pm(\mathrm{i}\hat{\partial}+\mathrm{i}m)\Phi_\pm(x)\Phi_\pm^\dagger(y)(-\mathrm{i}\overleftarrow{\hat{\partial}}+\mathrm{i}m) =\int\mathrm{d}^4z\mathrm{e}^{\mathrm{i}(k_2-k_1)z}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k_1x}\mathrm{e}^{-\mathrm{i}k_2y}N_\mathrm{c}^{-1} \times 2\pi\rho F(k_1)2\pi\rho F(k_2)\left(\frac{1\pm\gamma_5}{2}+\frac{\mathrm{i}m\hat{k}_2}{k_2^2}\frac{1\mp\gamma_5}{2} -\frac{\mathrm{i}m\hat{k}_1}{k_1^2}\frac{1\pm\gamma_5}{2}+\frac{m^2\hat{k}_1\hat{k}_2}{k_1^2k_2^2}\frac{1\mp\gamma_5}{2}\right). \quad (23)$$

The form-factor $F(k)$ is related to the zero-mode wave function in momentum space $\Phi_\pm(k;\xi_\pm)$ and is equal to

$$F(k)=-t\frac{\mathrm{d}}{\mathrm{d}t}\left[I_0(t)K_0(t)-I_1(t)K_1(t)\right],\quad t=\frac{1}{2}\sqrt{k^2}\rho. \quad (24)$$

At arbitrary N_f, a simple generalization of the approach in [5] to our case leads to

$$W_\pm=(-\mathrm{i})^{N_f}\left(\frac{4\pi^2\rho^2}{N_\mathrm{c}}\right)^{N_f}\int\frac{\mathrm{d}^4z}{V}\det{}_f(\mathrm{i}J_\pm(z)), \quad (25)$$

where we have defined

$$J_\pm(z)_{fg}=\int\frac{\mathrm{d}^4k_1\mathrm{d}^4k_2}{(2\pi)^8}\mathrm{e}^{\mathrm{i}(k_2-k_1)z}F(k_1)F(k_2)\psi_f^\dagger(k_1) \times\left(\frac{1\pm\gamma_5}{2}+\frac{\mathrm{i}m_g\hat{k}_2}{k_2^2}\frac{1\mp\gamma_5}{2}-\frac{\mathrm{i}m_f\hat{k}_1}{k_1^2}\frac{1\pm\gamma_5}{2} +\frac{m_fm_g\hat{k}_1\hat{k}_2}{k_1^2k_2^2}\frac{1\mp\gamma_5}{2}\right)\psi_g(k_2). \quad (26)$$

The two remarkable formulas

$$(ab)^N=\int\mathrm{d}\lambda\exp\left(N\ln\frac{aN}{\lambda}-N+\lambda b\right)\quad(N\gg1) \quad (27)$$

and

$$\exp(\lambda\det[\mathrm{i}A]) =\int\mathrm{d}\Phi\exp\left[-(N_f-1)\lambda^{-\frac{1}{N_f-1}}(\det\Phi)^{\frac{1}{N_f-1}}+\mathrm{i}\,\mathrm{tr}(\Phi A)\right] \quad (28)$$

have been used here. It is possible to check these formulas by the saddle-point approximation of the integrals. They were proposed in [5] and we followed this approach. Equation (27) leads to exponentiation, while (29) leads to the bosonization of the partition function (21). Starting from these formulas, we find

$$Z_N=\int\mathrm{d}\lambda_+\mathrm{d}\lambda_-\mathrm{D}\Phi_+\mathrm{D}\Phi_- \times\exp\left(-W[\lambda_+,\Phi_+;\lambda_-,\Phi_-]\right), \quad (29)$$

where

$$W\left[\lambda_+,\Phi_+;\lambda_-,\Phi_-\right] = -\sum_{\pm}\left(N_\pm \ln[\left(\frac{4\pi^2\rho^2}{N_\mathrm{c}}\right)^{N_f}\frac{N_\pm}{V\lambda_\pm}] - N_\pm\right) + w_\Phi + w_\psi, \quad (30)$$

$$w_\Phi = \int \mathrm{d}^4x \sum_\pm (N_f-1)\lambda_\pm^{-\frac{1}{N_f-1}}(\det\Phi_\pm)^{\frac{1}{N_f-1}},$$

$$w_\psi = -\mathrm{Tr}\ln\left(-\hat{k} + \mathrm{i}m_f + \mathrm{i}F(k_1)F(k_2)\sum_\pm \Phi_{\pm,gf}(k_1-k_2) \times\left(\frac{1\pm\gamma_5}{2} + \frac{\mathrm{i}m_g\hat{k}_2}{k_2^2}\frac{1\mp\gamma_5}{2} - \frac{\mathrm{i}m_f\hat{k}_1}{k_1^2}\frac{1\pm\gamma_5}{2} + \frac{m_f m_g\hat{k}_1\hat{k}_2}{k_1^2k_2^2}\frac{1\mp\gamma_5}{2}\right)(-\hat{k}+\mathrm{i}m_f)^{-1}\right).$$

Variation of the total action $W[\lambda_+,\Phi_+;\lambda_-\Phi_-]$ over $\lambda_\pm$, $\Phi_\pm$ must vanish in the common saddle point. In this point

$$\lambda_\pm = \lambda, \quad \Phi_{\pm,fg} = \Phi_{\pm,fg}(0) = M_f\delta_{fg}.$$

This condition leads to the definition of the momentum-dependent constituent mass $M_f(k)$, i. e.,

$$M_f(k) = M_f F^2(k). \quad (31)$$

The contribution of the quark loop to the saddle-point equation is

$$\mathrm{Tr}\ln\left[\left(-\hat{k} + \mathrm{i}m + \mathrm{i}F^2(k)\sum_\pm \Phi_\pm \times\left(\frac{1\pm\gamma_5}{2} + \frac{m^2}{k^2}\frac{1\mp\gamma_5}{2} \mp \frac{\mathrm{i}m\hat{k}}{k^2}\gamma_5\right)\right)(-\hat{k}+\mathrm{i}m)^{-1}\right]. \quad (32)$$

The details of the calculations of the Tr ln are

$$\mathrm{Tr}\ln\left[\left(-\hat{k}(1+\alpha\gamma_5) + \mathrm{i}\left(\Lambda_+\frac{1+\gamma_5}{2} + \Lambda_-\frac{1-\gamma_5}{2}\right)\right)/(-\hat{k})\right] = 2\ln\left(1-\alpha^2+\frac{\Lambda_+\Lambda_-}{k^2}\right), \quad (33)$$

where

$$\alpha = \frac{F^2(k)}{k^2}m(\Phi_+ - \Phi_-),$$

and

$$\Lambda_\pm = m + F^2(k)\left(\Phi_\pm + \Phi_\mp\frac{m^2}{k^2}\right).$$

Finally, the saddle-point equation is

$$\frac{4VN_\mathrm{c}}{N}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{M_f^2(k) + m_f M_f(k) + 2M_f^2(k)\frac{m_f^2}{k^2}}{k^2 + M_f^2(k) + 2m_f M_f(k) + 2M_f^2(k)\frac{m_f^2}{k^2}} = 1. \quad (34)$$

We only keep $O(m_f)$ terms and define $M_f = M_0 + \gamma m_f$. By expanding the left side of the saddle-point equation in m_f, we get

$$\frac{4VN_\mathrm{c}}{N}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{M_0^2F^4(k)}{k^2+M_0^2F^4(k)} = 1, \quad (35)$$

$$2\gamma\int k^2\mathrm{d}k^2\frac{k^2F^2(k)}{(k^2+M_0^2F^4(k))^2} = \int k^2\mathrm{d}k^2\frac{(M_0^2F^4(k)-k^2)F^2(k)}{(k^2+M_0^2F^4(k))^2}. \quad (36)$$

In [15] the simplified expression for the form-factor (24) was proposed:

$$F(k) = \frac{\Lambda^2}{\Lambda^2+k^2}, \quad (37)$$

where $\Lambda^2 \sim 2/\rho^2 = 0.72\,\mathrm{GeV}^2$.

The solution of the saddle-point equation (35) corresponding to $M_0 = 340\,\mathrm{MeV}$ demands that $\Lambda^2 = 0.76\,\mathrm{GeV}^2$. At $\Lambda^2 = 0.76\,\mathrm{GeV}^2$, $\gamma = -0.54$.

Finally, the constituent quark propagator has the form

$$S = (\mathrm{i}\hat{\partial} + \mathrm{i}(m_f + M_fF^2))^{-1}, \quad (38)$$

where M_f and F are given by (35), (36) and (37).

3 DP effective action

In order to clarify the important differences between the improved and the DP effective actions, we provide below the derivation of the DP effective action and propagator without giving all the details.

In order to find the DP effective action, we must start from the following representation of $\exp(\bar{\Omega}a^\mathrm{T}\Omega)$:

$$\exp(\bar{\Omega}a^\mathrm{T}\Omega) = \exp\int\left(\bar{\eta}(\mathrm{i}\hat{\partial})^{-1}\eta\right) = \left(\det(\mathrm{i}\hat{\partial})\right)^{-1}\int \mathrm{D}\psi\mathrm{D}\psi^\dagger\exp \times\int\mathrm{d}x\left(\psi^\dagger(x)\mathrm{i}\hat{\partial}\psi(x) - \bar{\eta}(x)\psi(x) + \psi^\dagger(x)\eta(x)\right), \quad (39)$$

where N_+, N_- sources $\eta = (\eta_+,\eta_-)$ and N_-, N_+ sources $\bar{\eta} = (\bar{\eta}_-,\bar{\eta}_+)$ are defined as

$$\bar{\eta}_- = \langle\mathrm{i}\hat{\partial}\Phi_{-,0}v_-|, \quad \bar{\eta}_+ = \langle\mathrm{i}\hat{\partial}\Phi_{+,0}u_+|,$$
$$\eta_+ = |\mathrm{i}\hat{\partial}\Phi_{+,0}\bar{u}_+\rangle, \quad \eta_- = |\mathrm{i}\hat{\partial}\Phi_{-,0}\bar{v}_-\rangle. \quad (40)$$

As shown in [10,12], this leads to:

$$\det{}_N = \int\mathrm{D}\psi\mathrm{D}\psi^\dagger\exp\left(\int\mathrm{d}^4x\sum_f\psi_f^\dagger\mathrm{i}\hat{\partial}\psi_f\right) \quad (41)$$
$$\times\prod_f\left\{\prod_+^{N_+}\left(\mathrm{i}m_f - V_+'[\psi_f^\dagger,\psi_f]\right)\prod_-^{N_-}\left(\mathrm{i}m_f - V_-'[\psi_f^\dagger,\psi_f]\right)\right\},$$

where

$$V'_{\pm}[\psi_f^\dagger,\psi_f]=\int \mathrm{d}^4x\left(\psi_f^\dagger(x)\mathrm{i}\hat\partial\Phi_{\pm,0}(x;\xi_\pm)\right)\times\int \mathrm{d}^4y\left(\Phi^\dagger_{\pm,0}(y;\xi_\pm)\mathrm{i}\hat\partial\psi_f(y)\right). \quad (42)$$

Equation (41) coincides with the ansatz for the fixed N partition function postulated by DP. This partition function leads to quite different equations for the effective mass M_f^{DP} and the propagator than (35), (36) and (38).

Effective masses M_f^{DP} now obey the equation

$$4N_\mathrm{c}V\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{M_f^2F^4(k)}{M_f^2F^4(k)+k^2}=N+\frac{m_fM_fVN_\mathrm{c}}{2\pi^2\rho^2}. \quad (43)$$

This also describes the shift of the effective mass of the quark M_f^{DP} due to current mass m_f. Expanding (43) with respect to m_f yields

$$M_f^{\mathrm{DP}}=M_0+\gamma_{\mathrm{DP}}m_f, \quad (44)$$

where

$$\gamma_{\mathrm{DP}}^{-1}=\rho^2\int_0^\infty \mathrm{d}k^2\frac{k^4F^4(k)}{(M_0^2F^4(k)+k^2)^2}. \quad (45)$$

Such integrals converge due to the form factor $F(k^2)$. Assuming for the parameters ρ and R the values (1) – which correspond to $M_0=340\,\mathrm{MeV}$ – we find

$$\gamma_{\mathrm{DP}}=2.4. \quad (46)$$

Also, with the DP effective action, one arrives at the following form for the constituent quark propagator:

$$S^{\mathrm{DP}}=(\mathrm{i}\hat\partial+\mathrm{i}M_f^{\mathrm{DP}}F^2)^{-1}, \quad (47)$$

where M_f^{DP} is given by (44) and (45).

4 Tests for the improved effective action

Improved and DP effective actions coincide in the chiral limit. Therefore we may expect essential differences in the results only beyond this limit.

We will test (30) by calculating the quark condensate, the masses of the pseudoscalar meson octet and axial-anomaly low-energy theorems, which will be reduced to the calculations of the specific correlators.

For these calculations we need an improved effective action in the presence of an external electromagnetic field a and of a field κ coupled with the topological density $g^2G\tilde G$. A similar problem was solved in our previous works [11,12], and we follow this method. The most essential part of the improved effective action, w_ψ, is transformed to

$$\begin{aligned}w_\psi[a,\kappa]=-\mathrm{Tr}\ln\Bigg(&-\hat K+\mathrm{i}m_f+\mathrm{i}F(k_1)F(k_2)\\&\sum_\pm\Phi_{\pm,gf}\left(1\pm(\kappa f)\right)^{N_f^{-1}}\Bigg(\frac{1\pm\gamma_5}{2}\\&+\frac{\mathrm{i}m_g\hat k_2}{k_2^2}\frac{1\mp\gamma_5}{2}-\frac{\mathrm{i}m_f\hat k_1}{k_1^2}\frac{1\pm\gamma_5}{2}\\&+\frac{m_fm_g\hat k_1\hat k_2}{k_1^2k_2^2}\frac{1\mp\gamma_5}{2}\Bigg)(-\hat k+\mathrm{i}m_f)^{-1}\Bigg). \quad (48)\end{aligned}$$

Here the gauged momentum is, as usual, $\hat K=\hat k-eQ\hat a$, the matrices $\Phi_\pm$, whose usual decomposition is

$$\Phi_\pm=\exp\left(\pm\frac{\mathrm{i}}{2}\phi\right)M\sigma\exp\left(\pm\frac{\mathrm{i}}{2}\phi\right),$$

where ϕ and σ are $N_f\times N_f$ matrices, describe mesons, and $M_{fg}=M_f\delta_{fg}$. At the saddle point $\sigma=1,\phi=0$. The usual decomposition for the pseudoscalar fields $\phi=\sum_0^8\lambda_i\phi_i$ may be used, where $\mathrm{Tr}\lambda_i\lambda_j=2\delta_{ij}$ and $\phi_{8(3)}$ can be identified with the $\eta(\pi^0)$-meson state.

4.1 The quark condensate and the pseudoscalar meson masses from improved effective action

First we calculate the quark condensate by using the evident formula

$$\begin{aligned}\mathrm{i}\langle\psi_f^\dagger\psi_f\rangle&=V^{-1}Z_N^{-1}\frac{\partial Z_N}{\partial m_f} \quad (49)\\&=-\frac{\delta W}{\delta m_f}=-\sum_\pm\left(\frac{\delta w_\Phi}{\delta\Phi_\pm}+\frac{\delta w_\psi}{\delta\Phi_\pm}\right)|_\Phi\frac{\delta\Phi}{\delta m_f}\\&+\mathrm{Tr}\,\mathrm{i}\left[(-\hat k+\mathrm{i}m_f+\mathrm{i}F^2M_f)^{-1}-(-\hat k+\mathrm{i}m_f)^{-1}\right].\end{aligned}$$

Another way is to calculate it directly:

$$\begin{aligned}\mathrm{i}\langle\psi_f^\dagger\psi_f\rangle=\mathrm{Tr}\,\mathrm{i}\Big[&(-\hat k+\mathrm{i}m_f+\mathrm{i}F^2M_f)^{-1}\\&-(-\hat k+\mathrm{i}m_f)^{-1}\Big]. \quad (50)\end{aligned}$$

The first term in (50) vanishes at the saddle point, and we have a perfect equivalence of the two calculations of the condensate, in contrast to analogous calculations with the DP action [5,11] (see also below). With the formula (50) we get

$$\mathrm{i}\langle\psi_f^\dagger\psi_f\rangle=N_\mathrm{c}\int\frac{k^2\mathrm{d}k^2}{4\pi^2}\times\left(\frac{m_f+M_fF^2(k)}{k^2+(m_f+M_fF^2(k))^2}-\frac{m_f}{k^2+m_f^2}\right). \quad (51)$$

Simple numerical calculations, using (35), (36) and (37), lead to

$$\begin{aligned}\mathrm{i}\langle\psi^\dagger\psi\rangle|_{m=0}&=0.0171\,\mathrm{GeV}^3, \quad (52)\\\mathrm{i}\langle\psi^\dagger\psi\rangle|_{m=0.15\,\mathrm{GeV}}&=0.0107\,\mathrm{GeV}^3,\\\frac{\langle\psi^\dagger\psi\rangle|_{m=0.15}}{\langle\psi^\dagger\psi\rangle|_{m=0}}-1&=-0.37.\end{aligned}$$

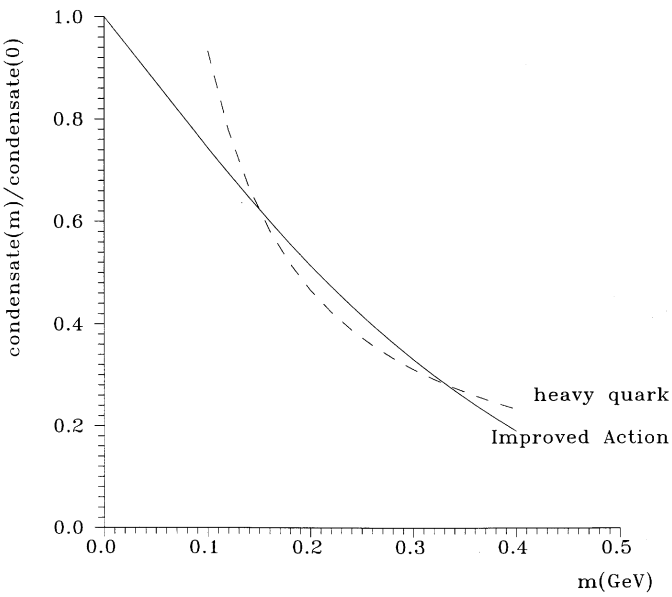

Fig. 1. The dependence of the condensate $\mathrm{i}\langle\psi^\dagger\psi\rangle$ on the current quark mass m normalized to the massless case. The solid line represents the result of the calculations with improved effective action (51). The dashed line represents results when using (53) for the heavy quark condensate.

More detailed information on the dependence of the condensate on the current quark mass is presented in Fig. 1. The solid line represents the result of the calculations using (51). It is clear from the figure that the calculations with the improved effective action lead to the expected dependence on the current mass [17]. The dashed line represents results when using the formula for the heavy quark condensate:

$$\mathrm{i}\langle\psi^\dagger\psi\rangle = \frac{g^2}{2^4\pi^2 3mV}\int \mathrm{d}x G^a_{\mu\nu}G^a_{\mu\nu} = \frac{2N}{3mV} = \frac{2}{3mR^4} = 0.0016m^{-1}. \tag{53}$$

This formula was derived in the lowest second order expansion over G/m^2 [18]. It is interesting that both curves (normalized to the massless case) almost coincide in the region of $m \sim 0.3\,\mathrm{GeV}$.

Now we calculate the masses of the pseudoscalar mesons. These mesons are considered to be a small fluctuation near the saddle point, where $\sigma = 1$. The term linear in ϕ in (30) is equal to zero at the saddle point, and we consider the next $O(\phi^2)$ terms. There is no contribution from w_Φ since $w_\Phi \sim (\prod_f M_f)^{1/(N_f-1)} = \mathrm{const}$. On the other hand, w_ψ makes a contribution like one-point and two-points diagrams:

$$\begin{aligned} w_\psi = &-\mathrm{Tr}\sum_f\Bigg[(-\hat{k}+\mathrm{i}(m_f+M_fF^2(k)))^{-1}F^2(k)\frac{(-\mathrm{i})}{8} \\ &\times(M\phi^2+\phi^2M+2\phi M\phi)_{ff}+\frac{1}{8} \\ &\times\sum_g((-\hat{k}_1+\mathrm{i}(m_f+M_fF^2(k_1)))^{-1} \\ &\times F(k_1)F(k_2)(M\phi(p)+\phi(p)M)_{fg} \\ &\times\gamma_5\left(1-\frac{\mathrm{i}m_g\hat{k}_2}{k_2^2}-\frac{\mathrm{i}m_f\hat{k}_1}{k_1^2}\right) \\ &\times(-\hat{k}_2+\mathrm{i}(m_g+M_gF^2(k_2)))^{-1} \\ &\times F(k_2)F(k_1)(M\phi(-p)+\phi(-p)M)_{gf} \\ &\times\gamma_5\left(1-\frac{\mathrm{i}m_g\hat{k}_2}{k_2^2}-\frac{\mathrm{i}m_f\hat{k}_1}{k_1^2}\right)\Bigg], \end{aligned} \tag{54}$$

where $p = k_1 - k_2$. First the $p = 0$-term is considered. From saddle-point equation (34) we get

$$\begin{aligned} w_\psi|_{p=0} = &-\frac{1}{8}\Bigg[\sum_f\frac{N}{VM_f}(M\phi^2+\phi^2M+2\phi M\phi)_{ff}) \\ &-\sum_g\Bigg(\frac{N}{VM_g^2}(M\phi+\phi M)^2_{gg}-\frac{m_g}{M_g}4N_\mathrm{c} \\ &\times\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{F^2(k)}{k^2+(m_f+M_fF^2(k))^2} \\ &\times(M\phi+\phi M)^2_{gg}\Bigg)\Bigg]+O(m^2) \\ = &\frac{1}{2}\sum_f m_f\mathrm{i}\langle\psi^\dagger\psi\rangle\phi^2_{ff} \qquad (55) \\ = &\frac{1}{2}\mathrm{i}\langle\psi^\dagger\psi\rangle(2m\sum_1^3\phi_i^2+(m_\mathrm{s}+m) \\ &\times\sum_4^7\phi_i^2+\frac{2}{3}(2m_\mathrm{s}+m)\phi_8^2)+O(m^2). \end{aligned}$$

Then

$$\frac{m_\mathrm{K}^2}{m_\pi^2} = \frac{m_\mathrm{s}+m}{2m} = 13.5$$

and

$$\frac{m_\eta^2}{m_\pi^2} = \frac{2m_\mathrm{s}+m}{3m} = 17.7,$$

where $m_\mathrm{u} = m_\mathrm{d} = m \sim 5\,\mathrm{MeV}$ and $m_\mathrm{s} \sim 130\,\mathrm{MeV}$ were used.

The experimental values of the masses lead to

$$\frac{m_\mathrm{K}^2}{m_\pi^2} = 13.4$$

and

$$\frac{m_\eta^2}{m_\pi^2} = 16.5.$$

The calculation of the p^2-term in w_ψ provides a normalization factor. Since the $p = 0$-term is on the order of m (and its $O(m^2)$ contributions were neglected), we calculate the p^2-term in the chiral limit $m = 0$. Then the p^2-term in w_ψ is extracted from

$$-\frac{1}{2}\mathrm{Tr}\sum_{fg}\phi_{fg}(p)\phi_{gf}(-p)M_0^2(-\hat{k}_1+\mathrm{i}M_0F^2(k_1))^{-1}$$

$$\times F(k_1)F(k_2)M_0^2\gamma_5(-\hat{k}_2+\mathrm{i}M_0F^2(k_2))^{-1}F(k_2)F(k_1)\gamma_5 = \frac{1}{2}\phi(p)\phi(-p)4N_c\int\frac{\mathrm{d}^4k}{(2\pi)^4}M_0^2F^2(k_2)F^2(k_1) \times\frac{k_1k_2+M_0^2F^2(k_1)F^2(k_2)}{(k_1^2+M_0^2F^4(k_1))(k_2^2+M_0^2F^4(k_2))}. \tag{56}$$

Accordingly [5], in the chiral limit:

$$4N_c\int\frac{\mathrm{d}^4k}{(2\pi)^4}M_0^2F^2(k_2)F^2(k_1) \times\frac{k_1k_2+M_0^2F^2(k_1)F^2(k_2)}{(k_1^2+M_0^2F^4(k_1))(k_2^2+M_0^2F^4(k_2))}=\frac{N}{V}-f_\pi^2p^2+\dots \tag{57}$$

Then, by combining this result with the calculations of the $p=0$-term, we get

$$m_\pi^2=\frac{\mathrm{i}\langle\psi^\dagger\psi\rangle 2m}{f_\pi^2}$$

and all the other masses of the octet of the pseudoscalar mesons. Therefore, the improved effective action successfully reproduces the current algebra results.

4.2 The quark condensate and the pseudoscalar meson masses from DP effective action

The two ways of calculating the quark condensate mentioned above lead to quite different formulas in the case of DP effective action:

$$\frac{1}{VZ_N^{\mathrm{DP}}}\frac{\mathrm{d}Z_N^{\mathrm{DP}}}{\mathrm{d}m_f}=\frac{N_cM_f^{\mathrm{DP}}}{2\pi^2\rho^2} \tag{58}$$

and

$$\mathrm{i}\langle\psi_f^\dagger\psi_f\rangle=4N_c\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{M_f^{\mathrm{DP}}F^2(k)}{M_f^{\mathrm{DP}2}F^4(k)+k^2}. \tag{59}$$

In the chiral limit (51) and (59) coincide and give a value which is almost the same as given in (58). In the case of the DP effective action, the current mass dependence of the quark condensate is induced by the m_f-dependence of the effective mass M_f^{DP}, which is given by (44) and (45). From (58) we get

$$\frac{1}{VZ_N^{\mathrm{DP}}}\frac{\mathrm{d}Z_N^{\mathrm{DP}}}{\mathrm{d}m}=(0.018+0.045m)\,\mathrm{GeV}^3, \tag{60}$$

and from (59) we get

$$\mathrm{i}\langle\psi^\dagger\psi\rangle=(0.016+0.181m)\,\mathrm{GeV}^3. \tag{61}$$

Therefore, both (60) and (61), derived from the DP effective action, give a completely wrong (and different) dependence on m beyond the chiral limit.

A similar calculation of the pseudoscalar meson masses as with the improved effective action (see (54)) and with the saddle-point equation (43) taken into account, leads to a $p=0$-term in w_ψ^{DP}:

$$w_\psi^{\mathrm{DP}}|_{p=0} = -\frac{1}{8}\sum_f\left[\frac{1}{M_f^{\mathrm{DP}}}(M^{\mathrm{DP}}\phi^2+\phi^2M^{\mathrm{DP}}+2\phi M^{\mathrm{DP}}\phi)_{ff} -\frac{1}{M_f^{\mathrm{DP}2}}(M^{\mathrm{DP}}\phi+\phi M^{\mathrm{DP}})^2_{ff}\right]\left(\frac{N}{V}+\frac{m_fM_f^{\mathrm{DP}}N_c}{2\pi^2\rho^2}\right) = -\frac{1}{4}\sum_{fg}\left(\frac{N}{V}+\frac{m_fM_f^{\mathrm{DP}}N_c}{2\pi^2\rho^2}\right)\phi_{fg}\frac{M_g^{\mathrm{DP}}}{M_f^{\mathrm{DP}}}\left(1-\frac{M_g^{\mathrm{DP}}}{M_f^{\mathrm{DP}}}\right) \times\phi_{gf}\sim O(m^2). \tag{62}$$

As can be seen, the DP effective action fails to reproduce the current algebra result for the pseudoscalar meson masses.

The next test is related to axial-anomaly low-energy theorems (LET) [16]. These theorems were used in [10,11] to check the DP effective action. The DP effective action was able to reproduce LET only in the chiral limit and failed beyond this limit.

LET1

The non-vanishing η' meson mass $m_{\eta'}$ implies, even in chiral limit (due to axial anomaly), that for real photons the matrix element of the divergence of the singlet axial current vanishes in the $q^2\ll m_{\eta'}^2$ limit, giving rise to the following low-energy theorem (LET1):

$$\langle 0|N_f\frac{g^2}{16\pi^2}G\tilde{G}|2\gamma\rangle+2\mathrm{i}\sum_f m_f\langle 0|\psi_f^\dagger\gamma_5\psi_f|2\gamma\rangle = N_c\frac{e^2}{4\pi^2}\sum_f Q_f^2F^{(1)}\tilde{F}^{(2)}, \tag{63}$$

where $F_{\mu\nu}^{(i)}=\epsilon_{i,\mu}q_{i,\nu}-\epsilon_{i,\nu}q_{i,\mu}$ and q_i,ϵ_i $(i=1,2)$ are the momentum and polarization vectors of photons and $q=q_1+q_2$, respectively. Equation (63) is an exact low-energy relation, which cannot be fulfilled in the framework of perturbation theory. Only a non-perturbative contribution of order g^{-2} – as the one provided by instantons – may cancel the factor g^2 at the first term of the l.h.s. The first term in the l.h.s. in (63) is calculated from a three-point correlator of the operator $g^2G\tilde{G}$ and two operators of the electromagnetic currents. It is clear from (48) that this correlator is equal to

$$\frac{\delta^3w_\psi[a,\kappa]}{\delta\kappa\delta a\delta a}|_{\Phi_\pm=M,a,\kappa=0}. \tag{64}$$

The operator $g^2G\tilde{G}$ generates the vertex $\mathrm{i}fF^2M_fN_f^{-1}\gamma_5$, where $f(q^2)$ is a momentum representation of the instanton contribution in the operator $g^2G\tilde{G}(x)$ and $f(0)=$

$32\pi^2$. At small q^2, this vertex is reduces to

$$32\pi^2 \mathrm{i} F^2 M_f N_f^{-1}\gamma_5. \tag{65}$$

Then the three-angular diagrams corresponding to the the anomaly contribution (the first term on the l.h.s. of (63)), with vertices (65), $eQ_f\gamma_\mu$ and propagator (38) lead to

$$2\mathrm{i}N_\mathrm{c}e^2Q_f^2F^{(1)}\tilde{F}^{(2)}\Gamma_f, \tag{66}$$

where Γ_f, the factor coming from the diagram of the process considered, may be calculated analytically if we approximate the form factor F by 1. In this approximation

$$\Gamma_f = \frac{M_f}{8\pi^2(M_f+m_f)}. \tag{67}$$

In the same approximation, the current mass contribution (the second term on the l.h.s. of (63)) leads to

$$2\mathrm{i}N_\mathrm{c}e^2Q_f^2F^{(1)}\tilde{F}^{(2)}\frac{m_f}{8\pi^2(M_f+m_f)}. \tag{68}$$

At the next step we combine (66) and (68) and sum over flavors. As a result, the l.h.s. and the r.h.s of (63) coincide. So the improved effective action immediately fulfills the low-energy theorem (63) even beyond the chiral limit, in contrast to the DP effective action result [11].

If we take the form factor F into account in (66) and (68) and give the model parameters the values (1), we find [10] a variation of ∼17%.

LET2, LET3

Further tests for the improved effective action can be obtained from the matrix elements of the divergence of the singlet axial current between vacuum and meson states. Neglecting $O(m^2)$ terms, we get the following equations:

$$\langle 0|N_f\frac{g^2}{16\pi^2}G\tilde{G}|\eta\rangle = -2\mathrm{i}m_\mathrm{s}\langle 0|\psi_\mathrm{s}^\dagger\gamma_5\psi_\mathrm{s}|\eta\rangle, \tag{69}$$

$$\langle 0|N_f\frac{g^2}{16\pi^2}G\tilde{G}|\pi^0\rangle = -\mathrm{i}(m_\mathrm{u}-m_\mathrm{d})\langle 0|\psi^\dagger\tau_3\gamma_5\psi|\pi^0\rangle, \tag{70}$$

which we call LET2 and LET3, respectively. These matrix elements are reduced to two-point correlators. It is rather easy to show that the improved effective action satisfies LET2 (69) and LET3 (70).

From previous considerations it follows that the factor $g^2G\tilde{G}$ generates the vertex $\mathrm{i}MfF^2\gamma_5N_f^{-1}$ and that the η-meson gives rise to $\mathrm{i}M_\mathrm{s}\lambda_8F^2\gamma_5$. The structure of the mass matrix M is

$$M = M_0+\gamma\left(m_\mathrm{s}\left(\frac{1}{3}-\frac{1}{\sqrt{3}}\lambda_8\right)+m_\mathrm{u}\frac{1+\tau_3}{2}+m_\mathrm{d}\frac{1-\tau_3}{2}\right). \tag{71}$$

Then at small q (and neglecting $m_\mathrm{u,d}$)

$$\begin{aligned}\langle 0|N_f\frac{g^2}{16\pi^2}G\tilde{G}|\eta\rangle \qquad\qquad (72)\\ = -\frac{16N_\mathrm{c}}{\sqrt{3}}\int\frac{\mathrm{d}^4k}{(2\pi)^4}F^4(k)\\ \times\left[\frac{M_\mathrm{s}^2}{((M_\mathrm{s}F^2(k)+m_\mathrm{s})^2+k^2)^2}-\frac{M_0^2}{(M_0^2F^4(k)+k^2)^2}\right].\end{aligned}$$

Expanding (72) over m_s, we get

$$\begin{aligned}\langle 0|N_f\frac{g^2}{16\pi^2}G\tilde{G}|\eta\rangle = -\frac{16N_\mathrm{c}m_\mathrm{s}}{\sqrt{3}}\int\frac{\mathrm{d}^4k}{(2\pi)^4}F^4(k)M_0\\ \times\frac{2\gamma k^2-2M_0^2F^2(k)}{(k^2+M_0^2F^4(k))^2}.\end{aligned} \tag{73}$$

From (36) we find, for the γ-factor, that

$$\begin{aligned}2\gamma\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2F^4(k)}{(k^2+M_0^2F^4(k))^2}\\ \approx\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(M_0^2F^4(k)-k^2)F^2(k)}{(k^2+M_0^2F^4(k))^2}.\end{aligned} \tag{74}$$

It is clear now that, by using (74), the l.h.s of (69) is reduced to the r.h.s. of (69), which is equal to

$$\begin{aligned}-2\mathrm{i}m_\mathrm{s}\langle 0|\psi_\mathrm{s}^\dagger\gamma_5\psi_\mathrm{s}|\eta\rangle\\ = \frac{16N_\mathrm{c}m_\mathrm{s}}{\sqrt{3}}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{M_0F^2(k)}{k^2+M_0^2F^4(k)}.\end{aligned} \tag{75}$$

The calculations with LET3 (70) are almost the same as with LET2 (69). Again, by using (74), the l.h.s. and the r.h.s. of (70) coincide. Hence the improved effective action satisfies LET2 and LET3, (69) and (70), respectively. For comparison, the DP effective action failed to reproduce these LET2 and LET3 [11].

Therefore, the improved effective action generates the correct current mass dependence of the vacuum quark condensate, reproduces current algebra results for the masses of the pseudoscalar meson octet, satisfies low-energy theorems LET2 and LET3 for the two-point correlators (69) and (70), respectively, and also satisfies LET1 for the three-point correlator (63), even beyond the chiral limit. We conclude that the improved effective action works properly beyond the chiral limit and provides the background for taking strange quarks into account.

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